PMM U.S.S.R.,Vol.47,No.1,pp.59-64,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 532.522.534.1

ON GENERATION OF AUTO-OSCILLATIONS DURING FIBERS FORMATION*

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The method of many scales is used to investigate the nonlinear interactions leading to appearance of auto-oscillations in the radius of the fiber during its forming. It is shown that the auto-oscillations are excited gradually and their amplitude and frequency is computed near the point of bifurcation. The behavior of the fiber is studied by numerical methods for the case when the draw-down ratio of spinning exceeds appreciably the critical value.

Experimental data /1,2/ show that the forming process loses its stability when the draw-down ratio attains some critical value, whereupon periodic oscillations in the values of the radius and velocity of the fiber appear. This phenomenon is called "draw resonance". In the linear approximation the steady state draw is shown /2 – 5/ to be unstable when the draw-down ratio exceeds the critical value of 20.22. Numerical computations /2/ have shown, in accordance with the experimental data, that the loss of stability is accompanied by the appearance of a cycle, i.e. of auto-oscillations.

1. Asymptotic investigations. In the process of drawing, liquid thread (fiber) is fed to spinnerets, thins during the motion, and is wound at some distance L from the spinneret onto the take-up bobbin. Under the model conditions the behavior of the liquid in the fiber corresponds to the rheological Newtonian relation. The motion in the liquid thread is best described in the quasi-one-dimensional approximation, neglecting the forces of inertia, weight, surface tension and friction against air (viscosity of the liquid is assumed sufficiently large to be the dominant factor). The equations of continuity and momentum have the form /6 – 8/

$$\frac{\partial a^2}{\partial t} + \frac{\partial a^2 V}{\partial x} = 0, \quad \frac{\partial}{\partial x}\left(a^2 \frac{\partial V}{\partial x}\right) = 0 \tag{1.1}$$

The following notation is used: t is time, x is the distance along the fiber counted from the spinneret orifice, a is the fiber radius and V is the longitudinal velocity along the fiber axis. In the general case the boundary conditions for the equations (1.1) are

$$x = 0, \ a = \zeta_1(t), \ V = \zeta_2(t); \ x = L, \ V = \zeta_3(t)$$

where $\zeta_i(t)$ are arbitrary functions of time. In other words, the radius and velocity of the fiber emerging from the spinneret and the rates of winding on the take-up bobbin can change with time according to an arbitrary law. Under the steady state conditions the above quantities become known constants $\zeta_1 = a^0$, $\zeta_2 = V^0$ and $\zeta_3 = V^1$. The draw-down ratio is $E = V^1/V^0$ and its critical value will be denoted by E_*. Choosing $L/V^1, L, a^0/\sqrt{E}, V^1$, as the scales for t, x, a and V, we retain the dimensionless equations of the problem in the form (1.1) and their stationary solution /6/, denoted by an upper bar, is written in the dimensionless form as

$$\bar{a} = E^{1/2(1-x)}, \quad \bar{V} = E^{(x-1)} \tag{1.2}$$

Let us introduce into our discussion the following perturbations of solution (1.2):

$$a = \bar{a}(1 + \varepsilon\alpha), \ V = \bar{V}(1 + \varepsilon\beta), \ \varepsilon \ll 1$$

The equations (1.1) now become

$$\frac{\partial\alpha}{\partial t} + \bar{V}\frac{\partial\alpha}{\partial x} + \frac{\bar{V}}{2}\frac{\partial\beta}{\partial x} + \varepsilon\bar{V}\left(\beta\frac{\partial\alpha}{\partial x} + \frac{\alpha}{2}\frac{\partial\beta}{\partial x}\right) = 0 \tag{1.3}$$
$$2\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial x} + \frac{1}{\ln E}\frac{\partial^2\beta}{\partial x^2} + \varepsilon\left(\alpha\frac{\partial\beta}{\partial x} + 2\beta\frac{\partial\alpha}{\partial x} + \frac{2}{\ln E}\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + \frac{1}{\ln E}\alpha\frac{\partial^2\beta}{\partial x^2}\right) = 0$$

Now the search for the auto-oscillations appearing as a result of the loss of stability by (1.2), reduces to study of the problem of eigenvalues of the system (1.3) with boundary conditions

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$$x = 0,\ \alpha = \beta = 0;\ x = 1,\ \beta = 0 \tag{1.4}$$

In this case the perturbations are introduced between the spinneret and the bobbin. When $t = 0$, the distributions of the radius and velocity along the fiber differ, for the given value of the draw-down ratio E, from the stationary values (1.2). A similar situation may occur e.g. when the rate of fiber take-up by the bobbin at $t = 0$ is increased. Solving the problem (1.3), (1.4) in the linear approximation, we arrive at the following characteristic equation for the spectrum of the eigenvalues λ (the solution is sought in the form $e^{\lambda t}F(x)$) /3- 5/:

$$-\int_0^1 \exp(\lambda E^{1-y}/\ln E)\,dy + \frac{\lambda}{\ln E}\int_0^1 \exp(\lambda E^{1-y}/\ln E)\,dy + \frac{1}{\ln E}\left[\exp\left(\frac{\lambda}{\ln E}\right) - \exp\left(\frac{\lambda E}{\ln E}\right)\right] = 0 \tag{1.5}$$

When $E < E_*$, all roots λ of this equation have negative real parts. When $E = E_* \approx 20.22$, equation (1.5) has two, single, purely imaginary roots $\lambda = \pm i\omega = \pm i\cdot 0.693$. Calculating with the help of (1.5) the quantity $\mathrm{Re}\{\lambda\}$ near the points $\lambda = \pm i\omega$, $E = E_*$ we find $\mathrm{Re}\{\lambda\} = 0.0053\,(E - E_*)$. Consequently, when $E > E_*$ the perturbations increase in the linear approximation without bounds and forming is unstable. We note that the characteristic equation (1.5) has, for $20.22 < E < 49.98$, only two roots with $\mathrm{Re}\{\lambda\} > 0$ and further increase in the value of E leads to the appearance of new pairs of roots λ with positive real parts.

To find the solutions of the problem (1.3), (1.4) bounded and not dying with time, we must take into account the nonlinear effects. In the nonlinear formulation the problem can be solved analytically near the critical value E_* when $(E - E_*)$ is small. In the case when $E \gtrsim E_*$, the spectrum of the linear problem (1.5) predicts a slow growth in the oscillation amplitude in accordance with the law $\exp\{O[(E - E_*)t]\}$ and the change in their phase by $O[(E - E_*)t]$. For the complex oscillation amplitude $A^{-1}dA/dt = O(E - E_*)$ is correspondingly small. For this reason, introducing the asymptotic series

$$E = E_*(1 + \varepsilon E_1 + \varepsilon^2 E_2 + \ldots),\ E_i = O(1) \tag{1.6}$$

we find that the solution in question varies not only in the "rapid" time scale t, but also in the scales of the set of "slow" times $T = \varepsilon t$, $T_1 = \varepsilon^2 t$, ... The weak nonlinearity of the system (1.3) leads to separation of the processes taking place in different time scales. In particular, the growth and nonlinear restriction of the oscillation amplitudes take place within the "slow" time scales. The method of many scales /9/ represents an adequate technique for obtaining the auto-oscillatory solutions of the nonlinear problem. The times $t, T, T_1, \ldots$ are regarded in this method as independent variables. The method reassembles the method of Krylov - Bogoliubov - Mitropol'skii /10/ in, that it enables us to obtain the amplitude and phase equations for the nonlinear oscillations in the case of a weak nonlinearity. We write the required functions in the form of asymptotic series

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots,\ \beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \ldots \tag{1.7}$$

The functions α_i and β_i are of the order of unity, and x, t, T, T_1 serve as the arguments of the functions α, β, α_i and β_i. The stationary distribution of the velocity $\bar V$ and complex $1/\ln E$ entering (1.3) can be written, with (1.6) taken into account, in the form

$$\begin{gathered}\frac{1}{\ln E} = S_* + \varepsilon S_1 + \varepsilon^2 S_2 + \ldots,\quad \bar V(x) = \bar V_*(x) + \varepsilon V_1(x) + \varepsilon^2 V_2(x) + \ldots \\ S_* = \frac{1}{\ln E_*},\quad S_1 = -\frac{E_1}{\ln^2 E_*},\quad S_2 = \frac{1}{\ln^2 E_*}\left[\frac{E_1^2}{\ln E_*} - (E_2 - E_1^2/2)\right] \\ \bar V_* = E_*^{(x-1)},\quad V_1 = \bar V_* E_1 (x - 1), \\ V_2 = \left[\frac{(1-x)(2-x)}{2}E_1^2 + E_2(x-1)\right]\bar V_*\end{gathered} \tag{1.8}$$

Passing in (1.3) to the "rapid" and "slow" time and taking into account (1.7) and (1.8), we obtain the equations of zero order in ε. For α_0 and β_0 these equations reduce /3/ to

$$\frac{\partial^2\alpha_0}{\partial x^2} + E_*^{(1-x)}\frac{\partial^2\alpha_0}{\partial x\,\partial t} = 0 \tag{1.9}$$

and the functions α_0, β_0 as well as $\alpha_i, \beta_i\ (i > 0)$ satisfy the boundary conditions (1.4). A solution of (1.9) corresponding to the sustained oscillation is found by separating the variables, and has the form

$$\alpha_0 = A(T, T_1) e^{i\omega t} X(x) + A^*(T, T_1) e^{-i\omega t} X^*(x) \quad (1.10)$$
$$X(x) = \mathrm{Ci}[g(x)] - \mathrm{Ci}(\gamma) + i\{\mathrm{Si}[g(x)] - \mathrm{Si}(\gamma)\}$$
$$g(x) = \omega S_* E_*^{1-x}, \ \gamma = g(0)$$

Here, in accordance with (1.5), we have $\omega = 0.693$. The asterisks denote the complex conjugate functions and the amplitude A is an arbitrary function of the "slow" times. The remaining solutions of (1.9) decay with time. The solution for β_0 corresponding to (1.10) has the form

$$\beta_0 = A e^{i\omega t} Y(x) + A^* e^{-i\omega t} Y^*(x) \quad (1.11)$$
$$Y(x) = -2X + 2S_* \left\{ \frac{i\omega}{\overline{V}_*} X - S_*^{-1} \exp[ig(x)] + S_*^{-1} \exp(i\gamma) \right\}$$

Having obtained from (1.3) the equations of first order in ε we transform them with help of (1.10) and (1.11) to the form

$$\frac{\partial^2 \alpha_1}{\partial x^2} + E_*^{(1-x)} \frac{\partial^2 \alpha_1}{\partial x \, \partial t} = Z e^{i\omega t} + Z^* e^{-i\omega t} + A^2 Z_1 e^{2i\omega t} + A^{*2} Z_1^* e^{-2i\omega t} + AA^* Z_2 \quad (1.12)$$
$$Z(x, T, T_1) = \left[-\frac{\partial A}{\partial T} + A i\omega E_1 (x - 1) \right] \frac{X'}{\overline{V}_*}$$
$$Z_1(x) = \frac{f_1(x)}{2S_*} - \frac{1}{\overline{V}_*} \Big[\overline{V}_*' \left(X'Y + \tfrac{1}{2} XY' \right) +$$
$$\overline{V}_* \left(X''Y + \tfrac{3}{2} X'Y' + \tfrac{1}{2} XY'' \right) \Big]$$
$$Z_2(x) = \frac{f_2(x)}{2S_*} - \frac{1}{\overline{V}_*} \Big\{ \overline{V}_*' f_3(x) + \overline{V}_* [X''Y^* + X'Y^{*\prime} + X^{*\prime\prime} Y +$$
$$X^{*\prime} Y' + \tfrac{1}{2} (X^{*\prime} Y' + X^* Y'' + X' Y^{*\prime} + X Y^{*\prime\prime})] \Big\}$$
$$f_1(x) = XY' + 2X'Y + S_* (2X'Y' + XY'')$$
$$f_2(x) = X^* Y' + X Y^{*\prime} + 2(X'Y^* + X^{*\prime} Y) +$$
$$S_* [2(X^{*\prime} Y' + X' Y^{*\prime}) + X^* Y'' + X Y^{*\prime\prime}]$$
$$f_3(x) = X'Y^* + X^{*\prime} Y + \tfrac{1}{2} (X^* Y' + X Y^{*\prime})$$

where the primes denote derivatives in x. We note that $Z_2(x)$ is a real function. Having obtained the solution of (1.12) and the corresponding β_1, and having satisfied the boundary conditions, we obtain

$$k_1 \frac{\partial A}{\partial T} + k_2 E_1 E_* A = 0 \quad (1.13)$$
$$k_1 = \left(S_* - \frac{1}{i\omega} \right) (e_1 - e_2) + S_* X(1) - S_* (e_1 - E_* e_2)$$
$$k_2 = \frac{1}{E_*} (-i\omega S_* + 1) [S_* e_1 + (1 - S_*) e_2] +$$
$$\frac{S_*}{E_*} (-2i\omega S_* + 1) X(1) + i\omega S_* [S_* e_1 / E_* + (1 - S_*) e_2] + S_* (e_1 - e_2)/E_*$$
$$e_1 = \exp(i\gamma / E_*), \ e_2 = \exp(i\gamma)$$

and here we find that

$$\alpha_1 = B(T, T_1) e^{i\omega t} X(x) + B^*(T, T_1) e^{-i\omega t} X^*(x) + \quad (1.14)$$
$$A^2 e^{2i\omega t} F_1(x) + A^{*2} e^{-2i\omega t} F_1^*(x) + AA^* F_2(x)$$
$$\beta_1 = B(T, T_1) e^{i\omega t} Y(x) + B^*(T, T_1) e^{-i\omega t} Y^*(x) +$$
$$A^2 e^{2i\omega t} Y_2(x) + A^{*2} e^{-2i\omega t} Y_2^*(x) + AA^* Y_3(x)$$

where B is an arbitrary function of the "slow" times. We also have

$$P(x) = -S_* \{\mathrm{Ci}[2g(x)] - \mathrm{Ci}(2\gamma) + \mathrm{Si}[2g(x)] - \mathrm{Si}(2\gamma)\} \quad (1.15)$$
$$b_1 = \Big\{ -2 \int_0^1 Z_1(\xi) P_\xi^{*\prime} [P(1) - P(\xi)] \, d\xi + 4i\omega S_* \int_0^1 Z_1(\xi) P_\xi^{*\prime} \times$$
$$[P(1) - P(\xi)] \, d\xi + 2S_* P'(1) \int_0^1 Z_1(\xi) P_\xi^{*\prime} \, d\xi +$$

$$2S_*\left[X'Y+\frac{1}{2}XY'\right]\Big|_{x=1}-\int_0^1 f_1(y)\,dy\Big\}\Big/\{2P(1)-4S_*i\omega P(1)\ -2S_*P'(1)-2S_*P'(0)\}$$

$$b_2=\frac{1}{2}\left[-2\int_0^1\int_0^y Z_2(\xi)\,d\xi\,dy+2S_*\int_0^1 Z_2(\xi)\,d\xi+2S_*f_3(1)\ -\int_0^1 f_2(y)\,dy\right]$$

$$F_1(x)=\int_0^x Z_1(\xi)P_\xi^{*\prime}\,[P(x)-P(\xi)]\,d\xi+b_1P(x)$$

$$F_2(x)=\int_0^x\int_0^y Z_2(\xi)\,d\xi\,dy+b_2x$$

$$Y_2(x)=-2F_1+\frac{4S_*i\omega}{\bar V_*}F_1+2S_*F_1'+2S_*\left(X'Y+\frac{1}{2}XY'\right)-\int_0^x f_1(y)\,dy-2S_*F_1'(0)$$

$$Y_3(x)=-2F_2+2S_*F_2'+2S_*f_3(x)-\int_0^x f_2(y)\,dy-2S_*F_2'(0)$$

We note that a prime accompanying the integrand functions with subscript ξ denotes their derivatives in ξ, while the number b_2 as well as the functions $F_2(x)$ and $Y_3(x)$ are all real. Computing with help of (1.13) the coefficients k_1 and k_2 we find $k_1=-2.34-i\cdot 5.37$, $k_2=0.155-i\cdot 0.034$, consequently function A determined by (1.13) will be given by

$$A\,(T,\,T_1)=C\,(T_1)\exp\,[(0.106-i\cdot 0.538)\,TE_1] \qquad (1.16)$$

where $(C\,(T_1)$ is an arbitrary function. The expression (1.16) describes, of course, the linear behavior of the spectrum (1.5) during the passage through the critical value $E=E_*$. Since we seek a bounded, auto-oscillating solution, it follows that E_1, with (1.16) taken into account, should be equal to zero. Thus the amplitude A of the auto-oscillating solution should depend only on T_1.

Separating from (1.3) equations of the order of ε^2, we transform them with help of (1.8), (1.10), (1.11) and (1.14) and $E_1=0$, to the form

$$\frac{\partial^2\alpha_2}{\partial x^2}+E_*^{(1-\varkappa)}\frac{\partial^2\alpha_2}{\partial x\,\partial t}=\varphi_1 e^{i\omega t}+\varphi_1^* e^{-i\omega t}+\psi \qquad (1.17)$$

$$\varphi_1(x,\,T,\,T_1)=\left[-\frac{\partial A}{\partial T_1}+Ai\omega E_2(x-1)-\frac{\partial B}{\partial T}\right]\frac{X'}{\bar V_*}+A^2A^*\varphi_2(x)$$

$$\varphi_2(x)=\frac{f_4(x)}{2S_*}+\frac{f_5(x)}{2}-\frac{1}{\bar V_*}\frac{\partial}{\partial x}\,[\bar V_*f_6(x)]$$

$$f_4(x)=X^*Y_2'+XY_3'+F_2Y'+F_1Y^{*\prime}+2F_2'Y+2F_1'Y^*+2X'Y_3+2X^{*\prime}Y_2$$

$$f_5(x)=2X'Y_3'+2X^{*\prime}Y_2'+2Y'F_2'+2F_1'Y^{*\prime}+XY_3''+X^*Y_2''+F_2Y''+F_1Y^{*\prime\prime}$$

$$f_6(x)=YF_2'+Y^*F_1'+Y_3X'+Y_2X^{*\prime}+{}^1\!/_2\,(XY_3'+X^*Y_2'+F_2Y'+F_1Y^{*\prime})$$

The function $\psi\,(x,\,t,\,T,\,T_1)$ includes the quadratic and cubic harmonics in t, , and a term independent of t. Let us now seek a undamped solution of (1.17) and the corresponding distribution of β_2 satisfying the boundary conditions. As a result we find that $\partial B/\partial T=0$ must hold and also

$$k_1\frac{dA}{dT_1}+k_2E_2E_*A+A^2A^*\times\left\{-W(1)+[i\omega W(1)+W'(1)]\,S_*+\frac{R}{2}\right\}=0 \qquad (1.18)$$

$$W(x)=S_*^2\int_0^x \varphi_2(\xi)\,X_\xi^{*\prime}\,[X(x)-X(\xi)]\,d\xi$$

$$R=2S_*f_6(1)-\int_0^1 [f_4(y)+S_*f_5(y)]\,dy$$

Introducing into our discussion the modulus and argument of the complex amplitude $A=\rho e^{i\nu}$ and calculating the coefficient accompanying the nonlinear term in (1.18), we obtain

$$\frac{d\rho}{dT_1} = 0.106E_2\rho - 0.143\rho^3, \quad \frac{d\nu}{dT_1} = -0.538E_2 - 1.721\rho^2 \tag{1.19}$$

We see that a normal bifurcation of the cycle formation takes place. The natural oscillations of the fiber radius ("stretch resonance") appear only in the case when $E > E_* \ (E_2 > 0)$, i.e. they are generated gradually. The relative perturbation in the fiber radius is $\alpha^0 = \varepsilon\alpha = 2\varepsilon\rho \operatorname{Re}\{Xe^{i(\omega t+\nu)}\}$, therefore

$$\alpha^0|_{x=1} = 2\varepsilon\rho\,[-0.705\cos(0.693t + \nu) + 1.39\sin(0.693t + \nu)] \tag{1.20}$$

When E is somewhat larger than E_*, then according to (1.19) and (1.20) a natural oscillation will establish itself in the finite cross section of the fiber, with the following characteristics:

$$(\alpha^0|_{x=1})_{\max} = 0.594\sqrt{E - E_*} = M, \quad (\alpha^0|_{x=1})_{\min} = -M \tag{1.21}$$

$$\omega^0 = 0.693 - 0.09\,(E - E_*)$$

where ω^0 is the angular frequency of the self-oscillations.

2. **Numerical computations.** We have also carried out a numerical solution of the initial system of equations of continuity and momentum for the perturbations $\alpha^0 = \varepsilon\alpha$ and $\beta^0 = \varepsilon\beta$ (see (1.3)). The derivatives in x were approximated in the equation of continuity on the lower time layer according to the corner scheme. This gave us the distribution of α^0 in x on the upper layer, after which the application of the double sweep method with help of the momentum equation gave the distribution of β^0 on the upper time layer. The distribution over x were approximated on the interval $0 \leqslant x \leqslant 1$ over 100 points. The initial and boundary conditions were chosen in the form

$$t = 0, \ \alpha^0 = 2\delta \operatorname{Re}\{X\}, \ \beta^0 = 2\delta \operatorname{Re}\{Y\}$$
$$x = 0, \ \alpha^0 = \beta^0 = 0; \ x = 1, \ \beta^0 = 0$$

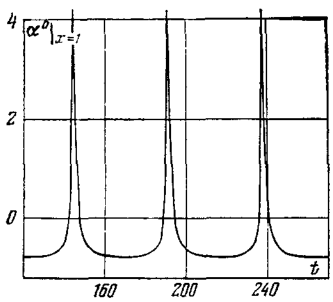

Fig.1

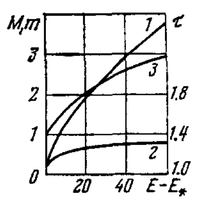

Fig.2

The computations have shown that in the finite difference approximation the critical value of the parameter E was slightly too high, $\simeq 25.5$ instead of the analytic result of 20.22. This makes the comparison of the numerical results with those of the asymptotic theory of Sect.1 near the critical value of E more difficult; only a qualitative comparison can be made. Irrespective of the amplitude of the initial perturbation (the values $0.01 \leqslant \delta \leqslant 0.6$ were used and we must have $\alpha^0 > -1$) at $E \lessapprox 25.5$ the oscillations were observed to dying out, while for $E > 25.5$ sustained oscillations were established with the amplitude independent of δ. The natural oscillations were excited in a mild manner, and this led to the normal bifurcation of the cylcle formation which agreed with the conclusions of Sect.1.

Finally we pause to look at the results of numerical study of the fiber behavior at the values of E exceeding appreciably that of E_*. The value of the draw-down ratio was increased up to the value $E = 700$. Sustained oscillations (i.e. a solution with limit cycle) correspond to the whole range under investigation, and the increase in E is accompanied by an increase in the period and amplitude of the oscillations. The results for $E = 95; \delta = 0.6$ are given in Fig.1 and Fig.2 shows the variation, with increasing $(E - E_*)$, of $M = (\alpha^0|_{x=1})_{\max}$ in curve 1 and $m = -(\alpha^0|_{x=1})_{\min}$ in curve 2, and the ratio τ of the period of natural oscillations to length of time $(E - 1)/\ln E$, in which the liquid particle travels under the steady state conditions from the spinneret to the bobbin by curve 3. The asymptotic behavior of the curves in Fig.2 when $(E - E_*) \to 0$ is described by the results (1.21).

The author thanks V.S. Berman and V.M. Entov for valuable comments.

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Translated by L.K.